Pade Approximations

## 4.6 Padé Approximations

In this section we introduce the notion of rational approximations for functions. The function $f(x)$ will be approximated over a small portion of its domain. For example, if $f(x) = \cos(x)$, it is sufficient to have a formula to generate approximations on the interval $[0, \pi/2]$. Then trigonometric identities can be used to compute $\cos(x)$ for any value $x$ that lies outside $[0, \pi/2]$.

A rational approximation to $f(x)$ on $[a, b]$ is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees $N$ and $M$, respectively. We use the notation $R_{N,M}(x)$ to denote this quotient:

$$R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)} \quad \text{for } a \leq x \leq b. \tag{1}$$

Our goal is to make the maximum error as small as possible. For a given amount of computational effort, one can usually construct a rational approximation that has a smaller overall error on $[a, b]$ than a polynomial approximation. Our development is an introduction and will be limited to Padé approximations.

The ***method of Padé*** requires that $f(x)$ and its derivative be continuous at $x = 0$. There are two reasons for the arbitrary choice of $x = 0$. First, it makes the manipulations simpler. Second, a change of variable can be used to shift the calculations over to an interval that contains zero. The polynomials used in (1) are

$$P_N(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_N x^N \tag{2}$$

and

$$Q_M(x) = 1 + q_1 x + q_2 x^2 + \cdots + q_M x^M. \tag{3}$$

The polynomials in (2) and (3) are constructed so that $f(x)$ and $R_{N,M}(x)$ agree at $x = 0$ and their derivatives up to $N + M$ agree at $x = 0$. In the case $Q_0(x) = 1$, the approximation is just the Maclaurin expansion for $f(x)$. For a fixed value of $N + M$ the error is smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $P_N(x)$ has degree one higher than $Q_M(x)$.

Notice that the constant coefficient of $Q_M$ is $q_0 = 1$. This is permissible, because it cannot be 0 and $R_{N,M}(x)$ is not changed when both $P_N(x)$ and $Q_M(x)$ are divided by the same constant. Hence the rational function $R_{N,M}(x)$ has $N + M + 1$ unknown coefficients. Assume that $f(x)$ is analytic and has the Maclaurin expansion

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots, \tag{4}$$

and form the difference $f(x)Q_M(x) - P_N(x) = Z(x)$:

$$\left(\sum_{j=0}^{\infty} a_j x^j\right)\left(\sum_{j=0}^{M} q_j x^j\right) - \sum_{j=0}^{N} p_j x^j = \sum_{j=N+M+1}^{\infty} c_j x^j. \tag{5}$$

The lower index $j = M + N + 1$ in the summation on the right side of (5) is chosen because the first $N + M$ derivatives of $f(x)$ and $R_{N,M}(x)$ are to agree at $x = 0$.

When the left side of (5) is multiplied out and the coefficients of the powers of $x^j$ are set equal to zero for $k = 0, 1, \ldots, N + M$, the result is a system of $N + M + 1$

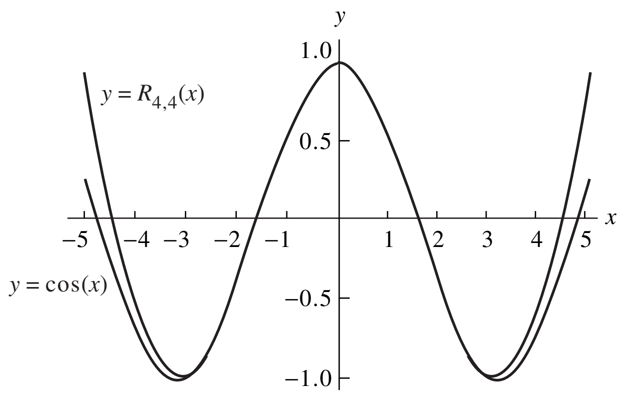


**Figure 4.18** The graph of $y = \cos(x)$ and its Padé approximation $R_{4,4}(x)$.

linear equations:

$$
\begin{aligned}
a_0 - p_0 &= 0 \\
q_1 a_0 + a_1 - p_1 &= 0 \\
q_2 a_0 + q_1 a_1 + a_2 - p_2 &= 0 \\
q_3 a_0 + q_2 a_1 + q_1 a_2 + a_3 - p_3 &= 0 \\
q_M a_{N-M} + q_{M-1} a_{N-M+1} + \cdots + a_N - p_N &= 0
\end{aligned}
\tag{6}
$$

and

$$
\begin{array}{llllll}
q_M a_{N-M+1} & + q_{M-1} a_{N-M+2} & + \cdots + q_1 a_N & & + a_{N+1} & = 0 \\
q_M a_{N-M+2} & + q_{M-1} a_{N-M+3} & + \cdots + q_1 a_{N+1} & & + a_{N+2} & = 0 \\
\vdots & & & & \vdots & \\
q_M a_N & + q_{M-1} a_{N+1} & + \cdots + q_1 a_{N+M-1} & & + a_{N+M} & = 0.
\end{array}
\tag{7}
$$

Notice that in each equation the sum of the subscripts on the factors of each product is the same, and this sum increases consecutively from 0 to $N + M$. The $M$ equations in (7) involve only the unknowns $q_1, q_2, \ldots, q_M$ and must be solved first. Then the equations in (6) are used successively to find $p_0, p_1, \ldots, p_N$.

**Example 4.17.** Establish the Padé approximation

$$
\cos(x) \approx R_{4,4}(x) = \frac{15{,}120 - 6900x^2 + 313x^4}{15{,}120 + 660x^2 + 13x^4}. \tag{8}
$$

See Figure 4.18 for the graphs of $\cos(x)$ and $R_{4,4}(x)$ over $[-5, 5]$.

If the Maclaurin expansion for $\cos(x)$ is used, we will obtain nine equations in nine unknowns. Instead, notice that both $\cos(x)$ and $R_{4,4}(x)$ are even functions and involve powers of $x^2$. We can simplify the computations if we start with $f(x) = \cos(x^{1/2})$:

$$f(x) = 1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \frac{1}{40{,}320}x^4 - \cdots . \tag{9}$$

In this case, equation (5) becomes

$$\left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \frac{1}{40{,}320}x^4 - \cdots\right)\left(1 + q_1x + q_2x^2\right) - p_0 - p_1x - p_2x^2$$
$$= 0 + 0x + 0x^2 + 0x^3 + 0x^4 + c_5x^5 + c_6x^6 + \cdots .$$

When the coefficients of the first five powers of $x$ are compared, we get the following system of linear equations:

$$\begin{aligned}
1 - p_0 &= 0\\
-\frac{1}{2} + q_1 - p_1 &= 0\\
\frac{1}{24} - \frac{1}{2}q_1 + q_2 - p_2 &= 0\\
-\frac{1}{720} + \frac{1}{24}q_1 - \frac{1}{2}q_2 &= 0\\
\frac{1}{40{,}320} - \frac{1}{720}q_1 + \frac{1}{24}q_2 &= 0.
\end{aligned} \tag{10}$$

The last two equations in (10) must be solved first. They can be rewritten in a form that is easy to solve:

$$q_1 - 12q_2 = \frac{1}{30} \qquad \text{and} \qquad -q_1 + 30q_2 = \frac{-1}{56}.$$

First find $q_2$ by adding the equations; then find $q_1$:

$$\begin{aligned}
q_2 &= \frac{1}{18}\left(\frac{1}{30} - \frac{1}{56}\right) = \frac{13}{15{,}120},\\
q_1 &= \frac{1}{30} + \frac{156}{15{,}120} = \frac{11}{252}.
\end{aligned} \tag{11}$$

Now the first three equations of (10) are used. It is obvious that $p_0 = 1$, and we can use $q_1$ and $q_2$ in (11) to solve for $p_1$ and $p_2$:

$$\begin{aligned}
p_1 &= -\frac{1}{2} + \frac{11}{252} = -\frac{115}{252},\\
p_2 &= \frac{1}{24} - \frac{11}{504} + \frac{13}{15{,}120} = \frac{313}{15{,}120}.
\end{aligned} \tag{12}$$

Now use the coefficients in (11) and (12) to form the rational approximation to $f(x)$:

$$f(x) \approx \frac{1 - 115x/252 + 313x^2/15{,}120}{1 + 11x/252 + 13x^2/15{,}120}. \tag{13}$$

Since $\cos(x) = f(x^2)$, we can substitute $x^2$ for $x$ in equation (13) and the result is the formula for $R_{4,4}(x)$ in (8). ■

## Continued Fraction Form

The Padé approximation $R_{4,4}(x)$ in Example 4.17 requires a minimum of 12 arithmetic operations to perform an evaluation. It is possible to reduce this number to seven by the use of continued fractions. This is accomplished by starting with (8) and finding the quotient and its polynomial remainder.

$$R_{4,4}(x) = \frac{15{,}120/313 - (6900/313)x^2 + x^4}{15{,}120/13 + (660/13)x^2 + x^4}$$
$$= \frac{313}{13} - \left(\frac{296{,}280}{169}\right)\left(\frac{12{,}600/823 + x^2}{15{,}120/13 + (600/13)x^2 + x^4}\right).$$

The process is carried out once more using the term in the previous remainder. The result is

$$R_{4,4}(x) = \frac{313}{13} - \cfrac{296{,}280/169}{\cfrac{15{,}120/13 + (660/13)x^2 + x^4}{12{,}600/823 + x^2}}$$
$$= \frac{313}{13} - \cfrac{296{,}280/169}{\cfrac{379{,}380}{10{,}699} + x^2 + \cfrac{420{,}078{,}960/677{,}329}{12{,}600/823 + x^2}}.$$

The fractions are converted to decimal form for computational purposes and we obtain

$$R_{4,4}(x) = 24.07692308 - \frac{1753.13609467}{35.45938873 + x^2 + 620.19928277/(15.30984204 + x^2)}. \tag{14}$$

To evaluate (14), first compute and store $x^2$, then proceed from the bottom right term in the denominator and tally the operations: addition, division, addition, addition, division, and subtraction. Hence it takes a total of seven arithmetic operations to evaluate $R_{4,4}(x)$ in continued fraction form in (14).

We can compare $R_{4,4}(x)$ with the Taylor polynomial $P_6(x)$ of degree $N = 6$, which requires seven arithmetic operations to evaluate when it is written in the nested form

$$P_6(x) = 1 + x^2\left(-\frac{1}{2} + x^2\left(\frac{1}{24} - \frac{1}{720}x^2\right)\right)$$
$$= 1 + x^2(-0.5 + x^2(0.0416666667 - 0.0013888889x^2)). \tag{15}$$

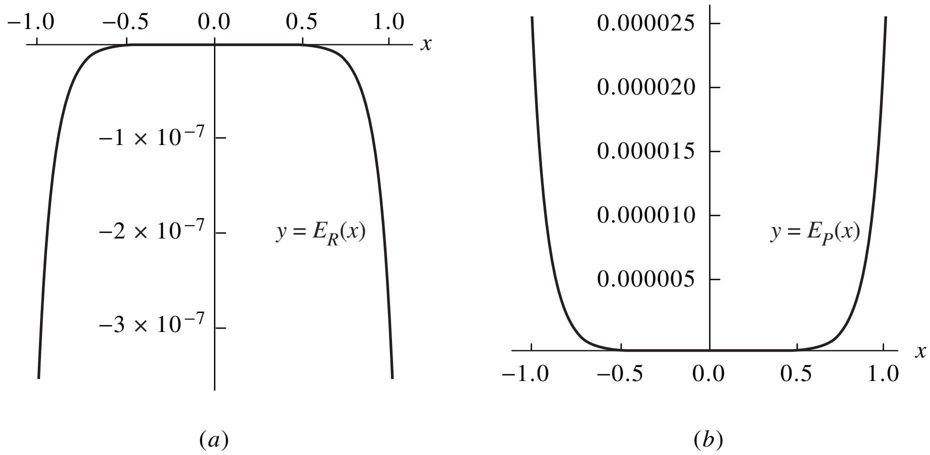


**Figure 4.19** (a) The graph of the error $E_R(x) = \cos(x) - R_{4,4}(x)$ for the Padé approximation $R_{4,4}(x)$. (b) The graph of the error $E_P(x) = \cos(x) - P_6(x)$ for the Taylor approximation $P_6(x)$.

The graphs of $E_R(x) = \cos(x) - R_{4,4}(x)$ and $E_P(x) = \cos(x) - P_6(x)$ over $[-1, 1]$ are shown in Figure 4.19(a) and (b), respectively. The largest errors occur at the endpoints and are $E_R(1) = -0.0000003599$ and $E_P(1) = 0.0000245281$, respectively. The magnitude of the largest error for $R_{4,4}(x)$ is about 1.467% of the error for $P_6(x)$. The Padé approximation outperforms the Taylor approximation better on smaller intervals, and over $[-0.1, 0.1]$ we find that $E_R(0.1) = -0.0000000004$ and $E_P(0.1) = 0.0000000966$, so the magnitude of the error for $R_{4,4}(x)$ is about 0.384% of the magnitude of the error for $P_6(x)$.

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